FINITE DIMENSIONAL ALGEBRAS ARISING FROM DIMER MODELS AND THEIR DERIVED EQUIVALENCES

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ABSTRACT. The notion of n-representation infinite algebra is a higher dimensional analogue of representation infinite hereditary algebra. It is known that this algebra can be obtained as the degree zero part of an $(n+1)$-Calabi-Yau algebra with a particularly nice grading. On the other hand, some 3-Calabi-Yau algebras are obtained from consistent dimer models which are bipartite graphs on the real-two torus. In this article, we first explain how to give a grading that induces a 2-representation infinite algebra to the 3-Calabi-Yau algebra arising from a consistent dimer model. Then, we study derived equivalent classes of 2-representation infinite algebras using perfect matchings of dimer models and their mutations.

Key Words: 2-representation infinite algebras, 3-Calabi-Yau algebras, Dimer models, Perfect matchings.

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1. Introduction

The notion of *n-representation infinite algebras* was introduced in [6] (see Definition 4 for the precise definition). That is a finite dimensional algebra having nice properties from the viewpoint of higher dimensional Auslander-Reiten theory, and it is an analogue of representation infinite hereditary algebras. For example, the Beilinson algebra which is arisen as the endomorphism ring of the tilting bundle $\bigoplus_{s=0}^{n}\mathcal{O}(s)$ on $\mathbb{P}^n$ is an n-representation infinite algebra (see [6, Example 2.15]). Also, it is known that this algebra is obtained as the degree zero part of a bimodule $(n+1)$-Calabi-Yau algebra of Gorenstein parameter 1 (see [4, 6, 11]). Some interesting examples of such a construction are given by dimer models as shown in [4, Section 6].

A *dimer model* (or *brane tiling*) Γ is a finite bipartite graph on the real two-torus $\mathbb{T}$, which induces a polygonal cell decomposition of $\mathbb{T}$. (When we consider the real two-torus $\mathbb{T}$, we fix the fundamental domain and identify $\mathbb{T}$ with $\mathbb{R}^2/\mathbb{Z}^2$.) That is, the set Γ_0 of nodes of Γ is divided into two parts Γ_0^+, Γ_0^-, and the set Γ_1 of edges consists of the ones connecting nodes in Γ_0^+ and those in Γ_0^-. In order to make the situation clear, we color nodes in Γ_0^+ white, and color nodes in Γ_0^- black. A connected component of $\mathbb{T}\backslash\Gamma_1$ is called a *face* of Γ, and we denote by Γ_2 the set of faces. In addition, in this article we assume that a dimer model satisfies the *consistency condition* (see e.g., [1, 7]), which is a certain nice condition on a dimer model.

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Then, as the dual of a dimer model Γ, we define the finite connected quiver $Q_\Gamma = Q$. That is, we assign a vertex of Q dual to each face in Γ_2, an arrow of Q dual to each edge in Γ_1, in which case we denote by Q_0 the set of vertices of Q and by Q_1 the set of arrows of Q. Here, the orientation of arrows is determined so that the white node is on the right of the arrow. For example, Figure 1 is a consistent dimer model and the associated quiver, where the outer frame is the fundamental domain of $\mathbb{T}$.

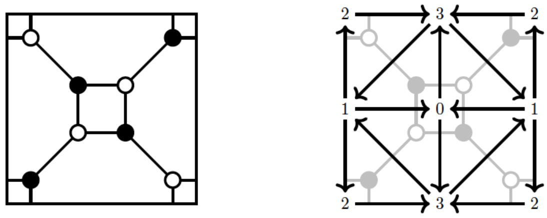

FIGURE 1. An example of a dimer model and the associated quiver

Let $\Bbbk$ be an algebraically closed field of characteristic zero. For the quiver Q associated with a dimer model, we consider the path algebra $\Bbbk Q$. Then, we define certain relations of Q as follows. By definition of the quiver Q, for each arrow $a \in Q_1$ there are precisely two oppositely oriented cycles containing the arrow a as a boundary. Let p_a^+ and p_a^- be the paths from $\mathsf{hd}(a)$ around the boundary of such cycles to $\mathsf{tl}(a)$. Here, $\mathsf{hd}, \mathsf{tl} : Q_1 \to Q_0$ are maps sending an arrow $a \in Q_1$ to the head of a and the tail of a respectively. Then, we consider the two-sided ideal $\mathcal{J}_Q = \langle p_a^+ - p_a^- \mid a \in Q_1 \rangle$ of $\Bbbk Q$ and set $A_Q = A := \Bbbk Q/\mathcal{J}_Q$. We call this algebra A the *Jacobian algebra* associated with a dimer model.

On the other hand, the notion of *perfect matching* plays a crucial role in this article.

Definition 1. A *perfect matching* (or *dimer configuration*) of a dimer model Γ is a subset D of Γ_1 such that for any node $n \in \Gamma_0$ there is a unique edge in D containing n as the end point. For a perfect matching D of Γ, we denote by D the subset of Q_1 obtained as the dual of D. We also say that D is a *perfect matching* of Q.

In general, every dimer model does not necessarily have a perfect matching. If a dimer model is consistent, then it has a perfect matching and every edge is contained in some perfect matchings (see e.g., [8, Proposition 8.1]). For example, Figure 2 shows perfect matchings of the dimer model Γ given in Figure 1.

For a perfect matching D of Q, we define the degree d_D on each arrow $a \in Q_1$ as

$$d_\mathsf{D}(a) = \begin{cases} 1 & \text{if } a \in \mathsf{D}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

This d_D induces the $\mathbb{Z}$-grading on the Jacobian algebra A, and we especially have the following.

Theorem 2 ([4, Proposition 6.1], see also [3])**.** *Let the notaiton be the same as above. Then, we see that A is a* bimodule 3-Calabi-Yau algebra of Gorenstein parameter 1*, that is, $A \in \mathrm{per} A^e$ and there exists a graded projective resolution $P_\bullet$ of A as A^e-module such that*

$$P_\bullet \cong P_\bullet^\vee[3](-1)$$

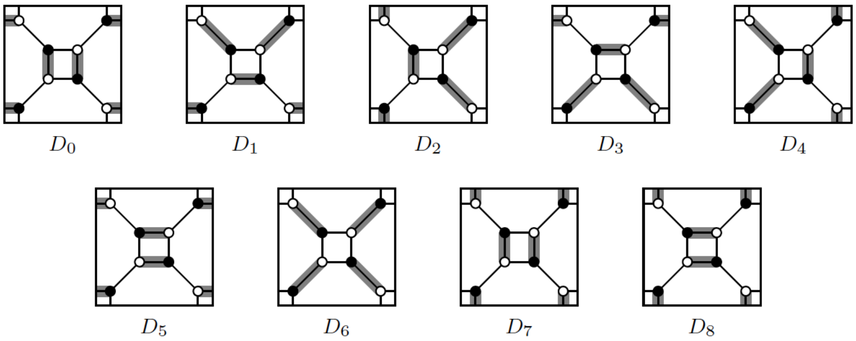

FIGURE 2. The perfect matchings of Γ

where $A^e := A \otimes_{\Bbbk} A^{\mathrm{op}}$ *and* $(-)^\vee := \mathrm{Hom}_{A^e}(-, A^e)$.

We define the *truncated Jacobian algebra*, which is denoted by A_{D}, as the degree zero part of the graded Jacobian algebra A with respect to d_{D}. The following is the motivating theorem in this article.

Theorem 3 (cf. [4, Corollary 3.6],[11, Theorem 4.12])**.** *Let A be the Jacobian algebra associated with a consistent dimer model, and D be a perfect matching of Q. Then, if the truncated Jacobian algebra A_{D} is finite dimensional, then it is a 2-representation infinite algebra, in which case the 3-preprojective algebra of A_{D} is A.*

Here, we recall the definition of n-representation infinite algebra for an integer $n > 0$.

Definition 4. We say that a finite dimensional algebra Λ is *n-representation infinite* if $\mathrm{gl.dim}\Lambda \leq n$ and $\nu_n^{-i}(\Lambda) \in \mathsf{mod}\Lambda$ for all $i \geq 0$ (this means $\nu_n^{-i}(\Lambda)$ is quasi-isomorphic to a complex concentrated in the degree zero part), in which case its global dimension is precisely n. Here, ν_n^- is the auto-equivalence on $\mathrm{D^b}(\mathsf{mod}\Lambda)$ defined by combining the Nakayama functor ν and the shift functor $[n]$, that is, $\nu_n^- := \nu^- \circ [n]$.

By Theorem 3, in order to construct 2-representation infinite algebras from the quiver associated with a consistent dimer model, we should understand the next question.

Question 5. *When is A_{D} finite dimensional ?*

2. On 2-representation infinite algebras arising from dimer models

In this section, we introduce the *perfect matching polygon* to answer Question 5. First, for each edge contained in a perfect matching of Γ, we give the orientation from a white node to a black node. We then fix a perfect matching D_0. For any perfect matching D, the difference of two perfect matchings $D - D_0$ forms a 1-cycle, and hence we consider it as the element in the homology group $\mathrm{H}_1(\mathbb{T}) \cong \mathbb{Z}^2$. When we consider $D - D_0$ as the element of $\mathrm{H}_1(\mathbb{T})$, we denote it by $[D - D_0]$. We then define the lattice polygon

$$\Delta_\Gamma := \mathrm{conv}\{[D - D_0] \in \mathbb{Z}^2 \mid D \text{ is a perfect matching of } \Gamma\}$$

as the convex hull of lattice points associated to perfect matchings. We call Δ_Γ the *perfect matching* (= *PM*) *polygon* (or *characteristic polygon*) of Γ. Although this lattice polygon depends on the fixed perfect matching, it is determined up to translations because

$$[D_i - D_j] = [D_i - D_k] - [D_j - D_k] \tag{2.1}$$

for any perfect matchings D_i, D_j, D_k.

Definition 6. Fix a perfect matching D_0. We say that a perfect matching D is

- a *corner* (or *extremal*) *perfect matching* if $[D - D_0]$ lies on a vertex of Δ_Γ,
- a *boundary* (or *external*) *perfect matching* if $[D - D_0]$ lies on a lattice point of the boundary of Δ_Γ,
- an *internal perfect matching* if $[D - D_0]$ lies on an interior lattice point of Δ_Γ.

We note that corner, boundary, and internal perfect matchings do not depend on a choice of the fixed one by (2.1). In addition, if a dimer model is consistent, then there exists a unique corner perfect matching corresponding to each vertex of Δ_Γ (see e.g., [3, Corollary 4.27], [8, Proposition 9.2]).

For example, we consider the dimer model Γ given in Figure 1, and fix the perfect matching D_0 (see Figure 2). Then, we can easily check that $[D_1 - D_0] = (1, 0)$, $[D_2 - D_0] = (0, 1)$, $[D_3 - D_0] = (-1, 0)$, $[D_4 - D_0] = (0, -1)$, and $[D_i - D_0] = (0, 0)$ for $i = 0, 5, 6, 7, 8$. Thus, we have the perfect matching polygon as shown in Figure 3. In particular, $D_1, \cdots, D_4$ are corner perfect matchings (and hence boundary ones), and $D_0, D_5, \cdots, D_8$ are internal ones.

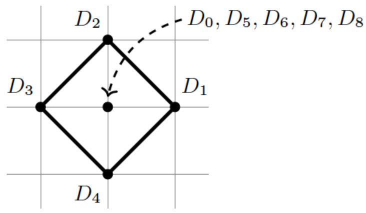

FIGURE 3. The perfect matching polygon of Γ

Then, we can obtain the next theorem, which was partially discussed in [2, Lemma 1.44].

Theorem 7 (see [12, Theorem 3.6])**.** *Let Q be the quiver associated with a consistent dimer model Γ. Then, for a perfect matching D of Q, the following conditions are equivalent.*

1. *D is an internal perfect matching.*
2. *Q_{D} is an acyclic quiver, where Q_{D} is the quiver obtained by deleting the arrows in D from Q.*
3. *A_{D} is a finite dimensional algebra.*

When this is the case, A_{D} is a 2-representation infinite algebra.

3. Mutations of perfect matchings

In the previous section, we saw that an internal perfect matching gives a 2-representation infinite algebra. In general, there are several internal perfect matching corresponding the same interior lattice point. Thus, in this section we will investigate the relationship between such internal perfect matchings.

First, we note that a perfect matching of Q can be considered as a *cut* in the sense of [5, 10], and the *mutation of cuts*, which was also introduced in [5, 10], is important to understand the relationship between cuts. In the following, we introduce this notion in terms of perfect matchings, and call it *the mutation of perfect matchings*.

Definition 8. We say that a vertex $k \in Q_0$ is a *strict source* (resp. *strict sink*) of (Q, D) if all arrows ending (resp. starting) at k belong to D and all arrows starting (resp. ending) at k do not belong to D. Namely, a strict source (resp. strict sink) is a source (resp. sink) of the quiver Q_{D}.

Definition 9. Let Q be the quiver associated with a dimer model, and D be a perfect matching of Q.

(1) We assume that $k \in Q_0$ is a strict source of (Q, D). We define a subset $\lambda_k^+(\mathsf{D})$ of Q_1 by removing all arrows in Q ending at k from D and adding all arrows in Q starting at k to D.
(2) Dually, we assume that $k \in Q_0$ is a strict sink of (Q, D), and define a subset $\lambda_k^-(\mathsf{D})$ of Q_1 by removing all arrows in Q starting at k from D and adding all arrows in Q ending at k to D.

The following properties follow from the definition.

Lemma 10. *Let D be a perfect matching of Q. For a strict source (resp. strict sink) $k \in Q_0$ of (Q, D), we have the followings.*

(a) $\lambda_k^+(\mathsf{D})$ (*resp.* $\lambda_k^-(\mathsf{D})$) *is a perfect matching of Q.*
(b) *k is a strict sink of $(Q, \lambda_k^+(\mathsf{D}))$ (resp. a strict source of $(Q, \lambda_k^-(\mathsf{D}))$).*
(c) *We have that $\lambda_k^-(\lambda_k^+(\mathsf{D})) = \mathsf{D}$ (resp. $\lambda_k^+(\lambda_k^-(\mathsf{D})) = \mathsf{D}$).*

Since $\lambda_k^{\pm}(\mathsf{D})$ are again perfect matchings, we call these operations the *mutations of a perfect matching* D of Q at $k \in Q_0$. We remark that since for an internal perfect matching D the quiver Q_{D} is acyclic (see Theorem 7), we can apply these mutations to D at some vertices. We also denote by $\lambda_k^+(D)$ (resp. $\lambda_k^-(D)$) the perfect matching of a dimer model Γ obtained as the dual of $\lambda_k^+(\mathsf{D})$ (resp. $\lambda_k^-(\mathsf{D})$), and call this the *mutation of a perfect matching D* of Γ at $k \in \Gamma_2$. We say that two perfect matchings are *mutation equivalent* if they are connected by repeating the mutations of perfect matchings.

The perfect matchings $D_0, D_5, \cdots, D_8$ given in Figure 2 are internal, and they correspond to the same interior lattice point (see Figure 3). By considering the mutations at appropriate faces, we see that these are mutation equivalent. This property holds for more general situation as follows.

Theorem 11 (see [12, Theorem 5.7])**.** *Let Γ be a consistent dimer model. Let D, D' be internal perfect matchings of Γ. Then, D and D' are mutation equivalent if and only if D and D' correspond to the same interior lattice point of the PM polygon of Γ.*

By combining this theorem with [10, Theorem 3.11], we have the following corollary.

Corollary 12. *Let Γ be a consistent dimer model, and Δ_Γ be the PM polygon of Γ. Let D_i, D_j be internal perfect matchings of Γ corresponding to the same interior lattice point of Δ_Γ. Then, we have an equivalence of derived categories $\mathrm{D^b}(\mathsf{mod} A_{\mathsf{D}_i}) \cong \mathrm{D^b}(\mathsf{mod} A_{\mathsf{D}_j})$.*

We note that this statement also follows from [9, Theorem 7.2 and Remark 7.3].

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